Rational angles in plane lattices

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Regular polygons with vertices on a plane lattice

Édouard Lucas, 1878

Les centres de trois cases quelconques d'un échiquier de grandeur quelconque ne sont jamais situés aux sommets d'un triangle équilatéral ou d'un hexagone régulier.

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Willy Scherrer, 1946

Es existiert kein einfaches oder sich selbst durchdringendes reguläres n-Eck, dessen Ecken sämtlich Punkte eines ebenen Gitters sind, ausgenommen in den Fällen $n = 3, 4$ *und* 6.

There is no regular n-agon (also self-intersecting) whose vertices all belong to a plane lattice, except for the cases $n = 3, 4, 6$.

The equation of two non-adjacent angles

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Two angles $\rightsquigarrow$ one equation
If $\theta, \mu \in U$, $\tau = \theta r$, $a, b \in \mathbb{Q}$, $ab(a-b) \neq 0$ and

$$\mu \frac{\tau + a}{\tau + b} \in \mathbb{R},$$

we obtain

$$r^2 + r\frac{a(y-x) + b(xy-1)}{\theta(y-1)} + ab = 0,$$

$\theta^2 = x, \mu^2 = y$.

Rational n-tuples

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We call *rational n-tuple* a set of n vectors such that each pair of them forms a rational angle.
We say that a space has type $\textcircled{n}$ if it contains a rational n-tuple; we extend this notation additively, saying that it has type $\textcircled{n}+\textcircled{m}$ if is contains a rational n-tuple and a disjoint rational m-tuple, and so on.

Three-angle equations

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Three-angle equations

Given two *independent* equations as before we can eliminate r

$$P_{④}(a, b, x, y, z) = (a - b)x + by - az - axy + bxz + (a - b)yz = 0.$$

$$\begin{aligned} P_{③+②}(a, b, c, x, y, z) = {} & (2a^2 - a(b + c) + 2bc)xy - abx^2y - acy - a(a - b)xy^2 \\ & - a(a - c)x + b(a - c)x^2 + c(a - b)y^2 \\ & - (2a^2 - a(b + c) + 2bc)xyz + acx^2yz \\ & + abyz + a(a - c)xy^2z + a(a - b)xz \\ & - c(a - b)x^2z - b(a - c)y^2z = 0. \end{aligned}$$

The equation of three non-adjacent angles

$$
\begin{aligned}
&P_{3\textcircled{2}}(a,b,c,d,x,y,z) = \\
&- b(b-c)c(a-d)y^2x^2 - b(a-c)(b-d)dy^2z^2x^2 - a(b-c)(a-d)dz^2x^2 + d\left(ba^2 + b^2a - 2bca - 2bda + cda + bcd\right)yz^2x^2 \\
&- a(a-c)c(b-d)x^2 + c\left(ba^2 + b^2a - 2bca - 2bda + cda + bcd\right)yx^2 + b\left(dc^2 + d^2c + abc - 2adc - 2bdc + abd\right)y^2zx^2 \\
&+ a\left(dc^2 + d^2c + abc - 2adc - 2bdc + abd\right)zx^2 + \left(-bca^2 - bda^2 - cd^2a - b^2ca - b^2da - c^2da + 8bcda - bcd^2 - bc^2d\right)yzx^2 \\
&+ (b-c)(a-d)(ab+cd)y^2x + (a-c)(b-d)(ab+cd)y^2z^2x + (b-c)(a-d)(ab+cd)z^2x \\
&+ \left(-2b^2a^2 + bca^2 + bda^2 - 2cda^2 + cd^2a + b^2ca + b^2da + c^2da - 2c^2d^2 + bcd^2 + bc^2d - 2b^2cd\right)yz^2x \\
&+ (a-c)(b-d)(ab+cd)x + \left(-2b^2a^2 + bca^2 + bda^2 - 2cda^2 + cd^2a + b^2ca + b^2da + c^2da - 2c^2d^2 + bcd^2 + bc^2d - 2b^2cd\right)yx \\
&+ \left(-2b^2a^2 + bca^2 + bda^2 - 2bc^2a - 2bd^2a + cd^2a + b^2ca + b^2da + c^2da - 2c^2d^2 + bcd^2 + bc^2d\right)y^2zx \\
&+ \left(-2b^2a^2 + bca^2 + bda^2 - 2bc^2a - 2bd^2a + cd^2a + b^2ca + b^2da + c^2da - 2c^2d^2 + bcd^2 + bc^2d\right)zx \\
&+ 2\left(2b^2a^2 - bca^2 - bda^2 + 2cda^2 + 2bc^2a + 2bd^2a - cd^2a - b^2ca - b^2da - c^2da - 4bcda + 2c^2d^2 - bcd^2 - bc^2d + 2b^2cd\right)yzx \\
&- a(b-c)(a-d)dy^2 - a(a-c)c(b-d)y^2z^2 - b(b-c)c(a-d)z^2 + c\left(ba^2 + b^2a - 2bca - 2bda + cda + bcd\right)yz^2 \\
&- b(a-c)(b-d)d + d\left(ba^2 + b^2a - 2bca - 2bda + cda + bcd\right)y + a\left(dc^2 + d^2c + abc - 2adc - 2bdc + abd\right)y^2z \\
&+ b\left(dc^2 + d^2c + abc - 2adc - 2bdc + abd\right)z + \left(-bca^2 - bda^2 - cd^2a - b^2ca - b^2da - c^2da + 8bcda - bcd^2 - bc^2d\right)yz = 0
\end{aligned}
$$

we seek solutions with x, y, z roots of unity different from 1 and a, b, c, d rational numbers, all distinct and different from 0.

When independent angles lead to proportional equations

If the two equations are not proportional, an algebraic solution gives a geometric configuration via

$$\tau = \frac{(cd-ab)x(y-1)(z-1)}{a(y-x)(z-1)+b(xy-1)(z-1)-c(z-x)(y-1)-d(xz-1)(y-1)}.$$

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If the two equations are not proportional, an algebraic solution gives a geometric configuration via

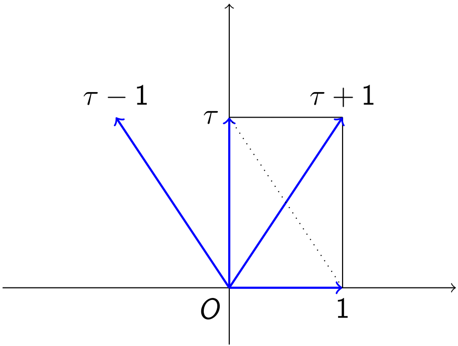
$$\tau = \frac{(cd - ab)x(y-1)(z-1)}{a(y-x)(z-1) + b(xy-1)(z-1) - c(z-x)(y-1) - d(xz-1)(y-1)}.$$

Otherwise, we have a simpler equation

$$\begin{aligned} P_{ab=cd} &= (d-b)xyz + (b-c)xy + (a-d)xz + (c-a)yz + \\ &+ (c-a)x + (a-d)y + (b-c)z + (d-b) = 0, \end{aligned}$$

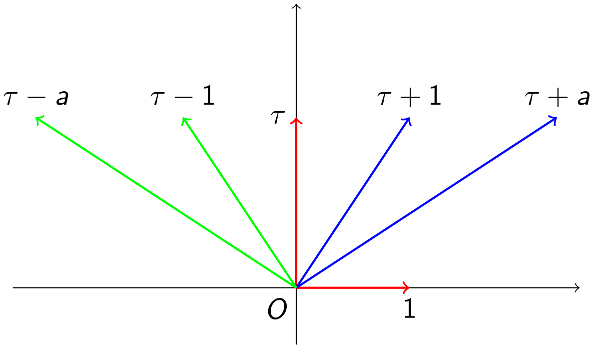
but in this case, unlike the others, an algebraic solution doesn't necessarily lead to a geometric configuration.

Families with unbounded order (1)



A family of type (4)

Families with unbounded order (2)



For every $a > 0$ in $\mathbb{Q}$, this is a family of type 3(2)

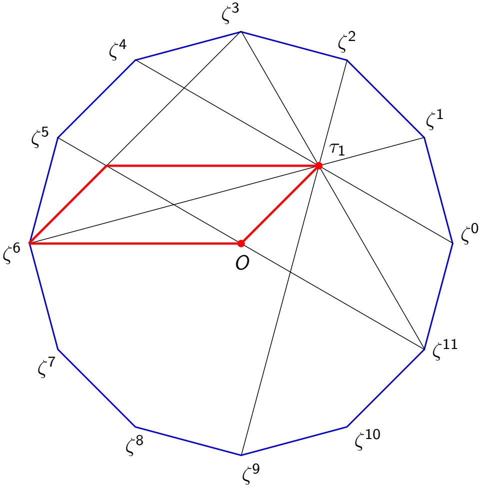
Spaces with infinitely many rational angles

The spaces $\mathbb{Q}(\sqrt{-d})$ with $d > 0$ squarefree integer are examples of infinitely many spaces, all not homotetic one another, each containing infinitely many rational angles.

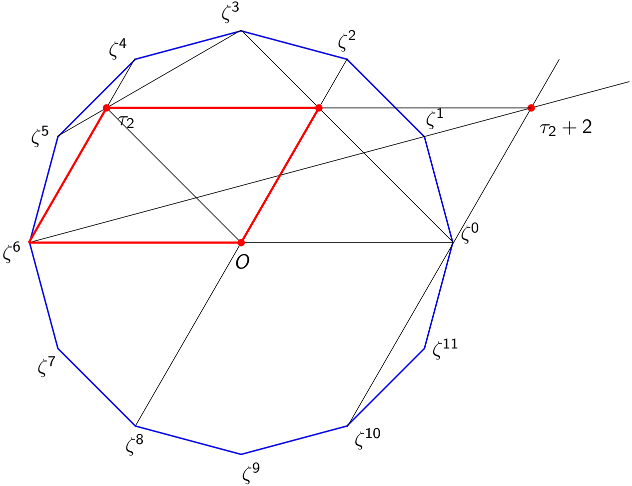
- for $d = 1$, $\mathbb{Q}(\sqrt{-d})$ has type ∞④
- for $d = 3$, $\mathbb{Q}(\sqrt{-d})$ has type ∞⑥
- for $d \neq 1, 3$, $\mathbb{Q}(\sqrt{-d})$ has type ∞②

Dodecagonal spaces

There are exactly two more spaces of type (4) up to homothety



dodecagonal spaces



Type ③+②

There are exactly four spaces of type 2③:

- $\langle \zeta_8^2 + \zeta_8 + 1, 1 \rangle_{\mathbb{Q}}$
- $\langle \zeta_5^2 + 2\zeta_5 + 2, 1 \rangle_{\mathbb{Q}}$
- $\langle 2\zeta_5^3 + \zeta_5 + 2, 1 \rangle_{\mathbb{Q}}$
- $\langle 2\zeta_{12}^3 + 2\zeta_{12}^2 - \zeta_{12} - 1, 1 \rangle_{\mathbb{Q}}$

they also correspond to intersection of diagonals in regular polygons.

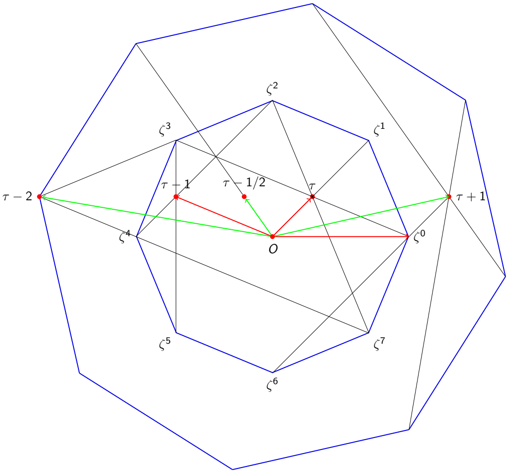


Figure: $\tau = \zeta_8^2 - \zeta_8 + 1$

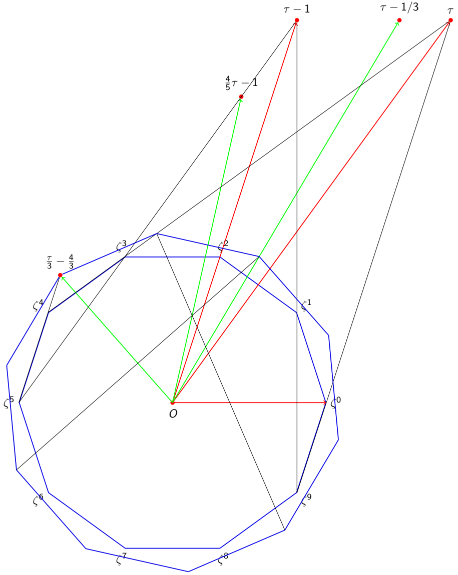


Figure: $\tau = \zeta_{10}^4 + 2\zeta_{10} + 2$

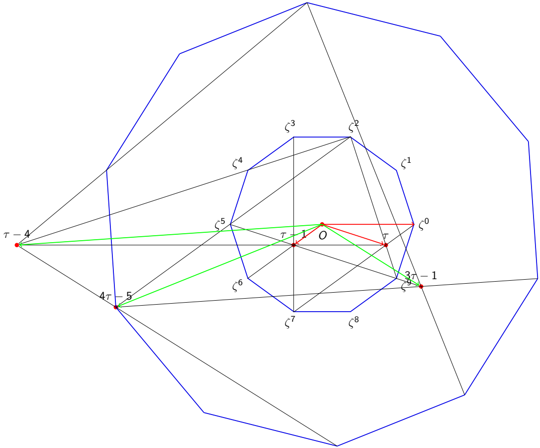


Figure: $\tau = \zeta_{10}^2 - 2\zeta_{10} + 2$

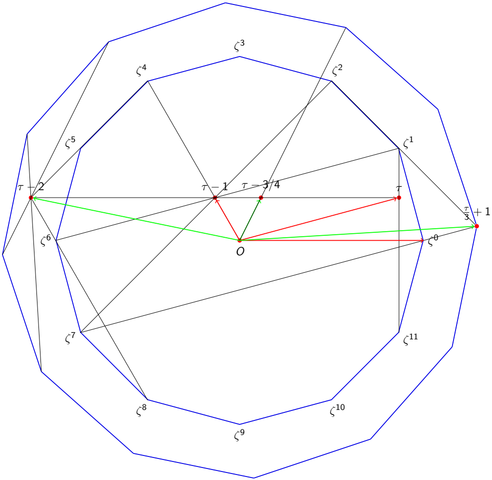


Figure: $\tau = -2\zeta_{12}^3 + 2\zeta_{12}^2 + \zeta_{12} - 1$

Case $ab = cd$

Two spaces of type 3$\textcircled{2}$ given by
$(a, b, c, d) = (3, \frac{1}{2}, \frac{3}{2}, 1)$

$$\tau_1 = -\frac{1}{2}\sqrt{\frac{1}{2}\left(2\sqrt{5} + \sqrt{52\sqrt{5}+45} + 13\right)} \cdot e^{\frac{3\pi i}{10}}, \quad \tau_2 = -\frac{1}{2}\sqrt{\frac{1}{2}\left(2\sqrt{5} + \sqrt{52\sqrt{5}+45} + 13\right)} \cdot e^{\frac{3\pi i}{10}}$$

$$\arg(\tau/\overline{\tau}) = \frac{3}{10}\cdot 2\pi, \quad \arg\left(\frac{\tau+a}{\tau+b} \Big/ \frac{\overline{\tau}+a}{\overline{\tau}+b}\right) = \frac{2}{5}\cdot 2\pi, \quad \arg\left(\frac{\tau+c}{\tau+d} \Big/ \frac{\overline{\tau}+c}{\overline{\tau}+d}\right) = \frac{1}{10}\cdot 2\pi,$$

and one rational families of type 5$\textcircled{2}$ involving sixth roots of unity plus an inequality condition. Within this subfamily there is exactly one more space of type 3$\textcircled{2}$

One rational family of type $4\textcircled{2}$ and angles $\zeta_8, \zeta_8, \zeta_8^3, \zeta_8^3$

One elliptic family of type $6\textcircled{2}$ and angles $\zeta_8, \zeta_8, \zeta_8^3, \zeta_8^3, \zeta_8^2, \zeta_8^2$

One elliptic family of type $6\textcircled{2}$ and angles $\zeta_8, \zeta_8, \zeta_8^3, \zeta_8^3 \zeta_8^4, \zeta_8^4$

One elliptic family of type $4\textcircled{2}$ and angles $\zeta_{10}, \zeta_{10}, \zeta_{10}^2, \zeta_{10}^2$

One elliptic family of type $4\textcircled{2}$ and angles $\zeta_{10}, \zeta_{10}, \zeta_{10}^3, \zeta_{10}^3$

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One elliptic family of type $4\textcircled{2}$ and angles $\zeta_{10}, \zeta_{10}, \zeta_{10}^5, \zeta_{10}^5$

One elliptic family of type $4\textcircled{2}$ and angles $\zeta_{10}^2, \zeta_{10}^2, \zeta_{10}^5, \zeta_{10}^5$

One rational family of type $4\textcircled{2}$ and angles $\zeta_{12}, \zeta_{12}, \zeta_{12}^5, \zeta_{12}^5$

One elliptic family of type $6\textcircled{2}$ and angles $\zeta_{12}, \zeta_{12}, \zeta_{12}^5, \zeta_{12}^5, \zeta_{12}^4, \zeta_{12}^4$

One elliptic family of type $6\textcircled{2}$ and angles $\zeta_{12}, \zeta_{12}, \zeta_{12}^5, \zeta_{12}^5, \zeta_{12}^6, \zeta_{12}^6$

One elliptic family of type $4\textcircled{2}$ and angles $\zeta_{12}^3, \zeta_{12}^3, \zeta_{12}^2, \zeta_{12}^2$

One elliptic family of type $4\textcircled{2}$ and angles $\zeta_{12}^3, \zeta_{12}^3, \zeta_{12}^4, \zeta_{12}^4$

Description of the familes in $\mathbb{Q}(\zeta_8)$

For $(\lambda, \mu) \in \mathbb{Q}^2 \setminus \{(0,0)\}$, let

$$\tau_{\lambda,\mu} := \lambda(\zeta_8^2 - \zeta_8^3) + \mu(1+\zeta_8) = \lambda \cdot \sqrt{2-\sqrt{2}}\zeta_{16} + \mu \cdot \sqrt{2+\sqrt{2}}\zeta_{16}.$$

The following hold:

- The space $\langle 1, \tau_{\lambda,\mu}\rangle_{\mathbb{Q}}$ only depends on $[\lambda : \mu] \in \mathbb{P}_1(\mathbb{Q})$. We denote it by $V_{[\lambda:\mu]}$.
- Let $\tau \in \mathbb{Q}(\zeta_8)$ generate a space V such that the squared angle between 1 and τ is ζ_8. There exists $(\lambda, \mu) \in \mathbb{Q}^2 \setminus \{(0,0)\}$ such that $\tau = \tau_{\lambda,\mu}$ and conversely, for every $(\lambda, \mu) \in \mathbb{Q}^2 \setminus \{(0,0)\}$, the squared angle between 1 and τ is ζ_8.
- For every $[\lambda : \mu] \in \mathbb{P}_1(\mathbb{Q})$, the complex conjugate $\overline{V_{[\lambda:\mu]}}$ is homothetic to $V_{[\lambda:\mu]}$.

Description of the familes in $\mathbb{Q}(\zeta_8)$

- The map

$$\begin{array}{rccc} f\colon & \mathbb{P}_1(\mathbb{Q}) & \to & \{\text{homothety classes of spaces}\} \\ & [\lambda:\mu] & \mapsto & \text{class of } V_{[\lambda:\mu]} \end{array}$$

  is 2-to-1. The involution $\iota : [\lambda : \mu] \mapsto [\lambda + \mu : \lambda - \mu]$ preserves the fibres of f.
- Let $S = \{[0:1],[1:0],[1:1],[-1:1]\}$. The spaces corresponding to $[\lambda:\mu] \in S$ have types $\textcircled{4}$ or $2\textcircled{3}$.

Description of the familes in $\mathbb{Q}(\zeta_8)$

- If $[\lambda:\mu] \notin S$, the space $V_{[\lambda:\mu]}$ contains precisely two angles with squared amplitude ζ_8 and precisely two angles with squared amplitude ζ_8^3. These are given respectively by

$$\zeta_8: \quad (1,\tau) \text{ and } (\tau+a,\tau+b),$$

where

$$a = \frac{\lambda^2-2\lambda\mu-\mu^2}{\lambda+\mu}, \quad b = -\frac{2\left(\lambda^3+2\lambda^2\mu-\lambda\mu^2\right)}{\lambda^2-2\lambda\mu-\mu^2}$$

and by

$$\zeta_8^3: \quad (\tau+c,\tau+d) \text{ and } (\tau+c',\tau+d')$$

where

$$c = \frac{-\lambda^2-2\lambda\mu+\mu^2}{2\lambda}, \quad d = -2(\lambda+\mu)$$

$$c' = -2\mu, \quad d' = \frac{\lambda^2+2\lambda\mu-\mu^2}{\mu-\lambda}.$$

Description of the familes in $\mathbb{Q}(\zeta_8)$

- Let $[\lambda : \mu] \notin S$. The space $V_{[\lambda:\mu]}$ contains an angle with squared amplitude ζ_8^2 if and only if $\lambda(\lambda^3 + \lambda^2\mu + \lambda\mu^2 + \mu^3)$ is the square of a rational number u. In this case, setting

$$e_\pm = \frac{\lambda^2 - \mu^2 \mp u}{\mu}, \quad f_\pm = \frac{-2\lambda\mu \pm u}{\lambda},$$

the space $V_{[\lambda:\mu]}$ contains precisely two angles with squared amplitude ζ_8^2, given by $(\tau + e_+, \tau + f_+)$ and $(\tau + e_-, \tau + f_-)$. The curve in weighted projective space

$$u^2 = \lambda(\lambda^3 + \lambda^2\mu + \lambda\mu^2 + \mu^3)$$

has genus 1 and a rational point, so it defines an elliptic curve. It is in particular isomorphic over $\mathbb{Q}$ to the elliptic curve with Weierstrass equation $y^2 = x^3 + x^2 + x + 1$, whose Mordell-Weil group over $\mathbb{Q}$ is $\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. A generator for the free part of the Mordell-Weil group is given by $(0, 1)$.

Description of the familes in $\mathbb{Q}(\zeta_8)$

- Let $[\lambda : \mu] \notin S$. The space $V_{[\lambda:\mu]}$ contains an angle with squared amplitude ζ_8^4 if and only if $3\lambda^4 + 2\lambda^3\mu + 4\lambda^2\mu^2 + 2\lambda\mu^3 + \mu^4$ is the square of a rational number v. In this case, setting

$$g = \frac{\lambda^2 - 2\lambda\mu - \mu^2 \mp v}{\lambda + \mu}, \quad h = \frac{\lambda^2 - 2\lambda\mu - \mu^2 \pm v}{\lambda + \mu},$$

  the space $V_{[\lambda:\mu]}$ contains precisely two angles with squared amplitude ζ_8^4 are precisely $(\tau + g_+, \tau + h_+)$ and $(\tau + g_-, \tau + h_-)$. The curve in weighted projective space

$$v^2 = 3\lambda^4 + 2\lambda^3\mu + 4\lambda^2\mu^2 + 2\lambda\mu^3 + \mu^4$$

  has genus 1 and a rational point, so it defines an elliptic curve. It is in particular isomorphic over $\mathbb{Q}$ to the elliptic curve with Weierstrass equation $y^2 + 2xy - 2y = x^3 - 6x^2 - 3x$, whose Mordell-Weil group over $\mathbb{Q}$ is $\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. A generator for the free part of the Mordell-Weil group is given by $(0, 2)$.
- Let $[\lambda : \mu] \notin S$. The space $V_{[\lambda:\mu]}$ does not have both an angle with squared amplitude ζ_8^2 and an angle with squared amplitude ζ_8^4.