

Rational angles in plane lattices

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Regular polygons with vertices on a plane lattice

Édouard Lucas, 1878

Les centres de trois cases quelconques d'un échiquier de grandeur quelconque ne sont jamais situés aux sommets d'un triangle équilatéral ou d'un hexagone régulier.

Three elements of $\mathbb{Z}[i]$ cannot be located at the vertices of a regular triangle or of a regular hexagon.

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Willy Scherrer, 1946

Es existiert kein einfaches oder sich selbst durchdringendes reguläres n -Eck, dessen Ecken sämtlich Punkte eines ebenen Gitters sind, ausgenommen in den Fällen $n = 3, 4$ und 6 .

There is no regular n -agon (also self-intersecting) whose vertices all belong to a plane lattice, except for the cases $n = 3, 4, 6$.

The equation of two non-adjacent angles

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Two angles \leadsto one equation

If $\theta, \mu \in U$, $\tau = \theta r$, $a, b \in \mathbb{Q}$, $ab(a - b) \neq 0$ and

$$\mu \frac{\tau + a}{\tau + b} \in \mathbb{R},$$

we obtain

$$r^2 + r \frac{a(y - x) + b(xy - 1)}{\theta(y - 1)} + ab = 0,$$

$$\theta^2 = x, \mu^2 = y.$$

Rational n -tuples

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We say that a space has type \textcircled{n} if it contains a rational n -tuple; we extend this notation additively, saying that it has type $\textcircled{n} + \textcircled{m}$ if it contains a rational n -tuple and a disjoint rational m -tuple, and so on.

Three-angle equations

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Three-angle equations

Given two *independent* equations as before we can eliminate r

$$P_{\textcircled{4}}(a, b, x, y, z) = (a - b)x + by - az - axy + bxz + (a - b)yz = 0.$$

$$\begin{aligned} P_{\textcircled{3}+\textcircled{2}}(a, b, c, x, y, z) &= (2a^2 - a(b + c) + 2bc)xy - abx^2y - acy - a(a - b)xy^2 \\ &\quad - a(a - c)x + b(a - c)x^2 + c(a - b)y^2 \\ &\quad - (2a^2 - a(b + c) + 2bc)xyz + acx^2yz \\ &\quad + abyz + a(a - c)xy^2z + a(a - b)xz \\ &\quad - c(a - b)x^2z - b(a - c)y^2z = 0. \end{aligned}$$

The equation of three non-adjacent angles

$$\begin{aligned} P_3 \textcircled{2}(a, b, c, d, x, y, z) = & -b(b-c)c(a-d)y^2x^2 - b(a-c)(b-d)dy^2z^2x^2 - a(b-c)(a-d)dz^2x^2 + d(ba^2 + b^2a - 2bca - 2bda + cda + bcd)yz^2x^2 \\ & - a(a-c)c(b-d)x^2 + c(ba^2 + b^2a - 2bca - 2bda + cda + bcd)yx^2 + b(dc^2 + d^2c + abc - 2adc - 2bdc + abd)y^2zx^2 \\ & + a(dc^2 + d^2c + abc - 2adc - 2bdc + abd)zx^2 + (-bca^2 - bda^2 - cd^2a - b^2ca - b^2da - c^2da + 8bcda - bcd^2 - bc^2d)yzx^2 \\ & + (b-c)(a-d)(ab+cd)y^2x + (a-c)(b-d)(ab+cd)y^2z^2x + (b-c)(a-d)(ab+cd)z^2x \\ & + (-2b^2a^2 + bca^2 + bda^2 - 2cda^2 + cd^2a + b^2ca + b^2da + c^2da - 2c^2d^2 + bcd^2 + bc^2d - 2b^2cd)yz^2x \\ & + (a-c)(b-d)(ab+cd)x + (-2b^2a^2 + bca^2 + bda^2 - 2cda^2 + cd^2a + b^2ca + b^2da + c^2da - 2c^2d^2 + bcd^2 + bc^2d - 2b^2cd)yx \\ & + (-2b^2a^2 + bca^2 + bda^2 - 2bc^2a - 2bd^2a + cd^2a + b^2ca + b^2da + c^2da - 2c^2d^2 + bcd^2 + bc^2d)y^2zx \\ & + (-2b^2a^2 + bca^2 + bda^2 - 2bc^2a - 2bd^2a + cd^2a + b^2ca + b^2da + c^2da - 2c^2d^2 + bcd^2 + bc^2d)zx \\ & + 2(2b^2a^2 - bca^2 - bda^2 + 2cda^2 + 2bc^2a + 2bd^2a - cd^2a - b^2ca - b^2da - c^2da - 4bcda + 2c^2d^2 - bcd^2 - bc^2d + 2b^2cd)yzx \\ & - a(b-c)(a-d)dy^2 - a(a-c)c(b-d)y^2z^2 - b(b-c)c(a-d)z^2 + c(ba^2 + b^2a - 2bca - 2bda + cda + bcd)yz^2 \\ & - b(a-c)(b-d)d + d(ba^2 + b^2a - 2bca - 2bda + cda + bcd)y + a(dc^2 + d^2c + abc - 2adc - 2bdc + abd)y^2z \\ & + b(dc^2 + d^2c + abc - 2adc - 2bdc + abd)z + (-bca^2 - bda^2 - cd^2a - b^2ca - b^2da - c^2da + 8bcda - bcd^2 - bc^2d)yz = 0 \end{aligned}$$

we seek solutions with x, y, z roots of unity different from 1 and a, b, c, d rational numbers, all distinct and different from 0.

When independent angles lead to proportional equations

If the two equations are not proportional, an algebraic solution gives a geometric configuration via

$$\tau = \frac{(cd - ab)x(y - 1)(z - 1)}{a(y - x)(z - 1) + b(xy - 1)(z - 1) - c(z - x)(y - 1) - d(xz - 1)(y - 1)}.$$

When independent angles lead to proportional equations

If the two equations are not proportional, an algebraic solution gives a geometric configuration via

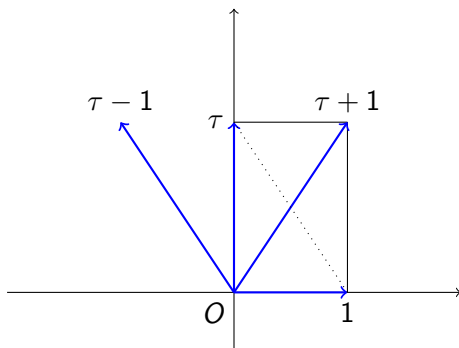
$$\tau = \frac{(cd - ab)x(y - 1)(z - 1)}{a(y - x)(z - 1) + b(xy - 1)(z - 1) - c(z - x)(y - 1) - d(xz - 1)(y - 1)}.$$

Otherwise, we have a simpler equation

$$P_{ab=cd} = (d - b)xyz + (b - c)xy + (a - d)xz + (c - a)yz + (c - a)x + (a - d)y + (b - c)z + (d - b) = 0,$$

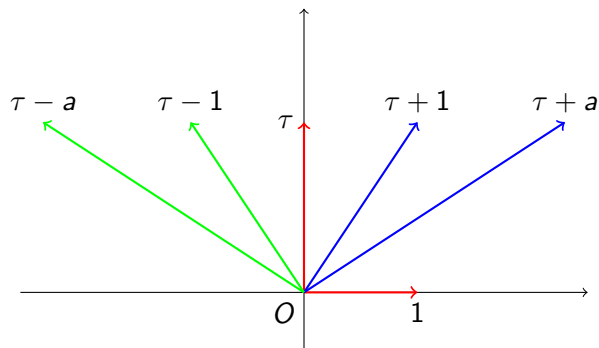
but in this case, unlike the others, an algebraic solution doesn't necessarily lead to a geometric configuration.

Families with unbounded order (1)



A family of type (4)

Families with unbounded order (2)



For every $a > 0$ in \mathbb{Q} , this is a family of type 3 (2)

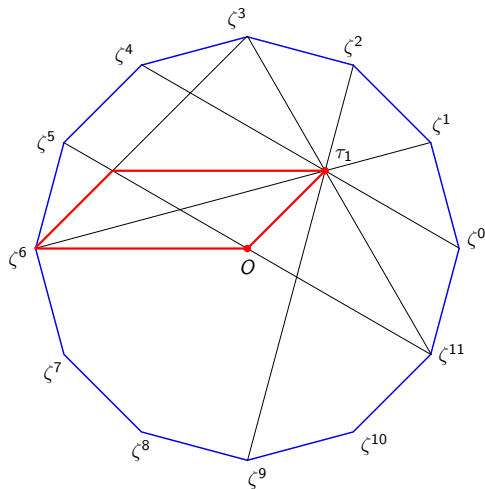
Spaces with infinitely many rational angles

The spaces $\mathbb{Q}(\sqrt{-d})$ with $d > 0$ squarefree integer are examples of infinitely many spaces, all not homotetic one another, each containing infinitely many rational angles.

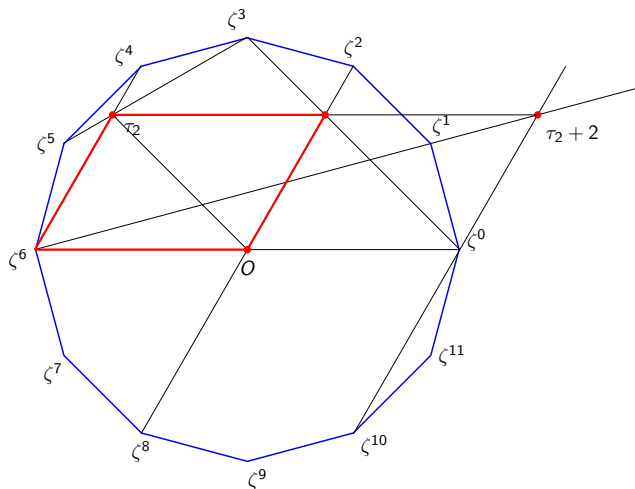
- for $d = 1$, $\mathbb{Q}(\sqrt{-d})$ has type $\infty(4)$
- for $d = 3$, $\mathbb{Q}(\sqrt{-d})$ has type $\infty(6)$
- for $d \neq 1, 3$, $\mathbb{Q}(\sqrt{-d})$ has type $\infty(2)$

Dodecagonal spaces

There are exactly two more spaces of type ④ up to homothety



dodecagonal spaces



Type $\textcircled{3} + \textcircled{2}$

There are exactly four spaces of type $2\textcircled{3}$:

- $\langle \zeta_8^2 + \zeta_8 + 1, 1 \rangle_{\mathbb{Q}}$
- $\langle \zeta_5^2 + 2\zeta_5 + 2, 1 \rangle_{\mathbb{Q}}$
- $\langle 2\zeta_5^3 + \zeta_5 + 2, 1 \rangle_{\mathbb{Q}}$
- $\langle 2\zeta_{12}^3 + 2\zeta_{12}^2 - \zeta_{12} - 1, 1 \rangle_{\mathbb{Q}}$

they also correspond to intersection of diagonals in regular polygons.

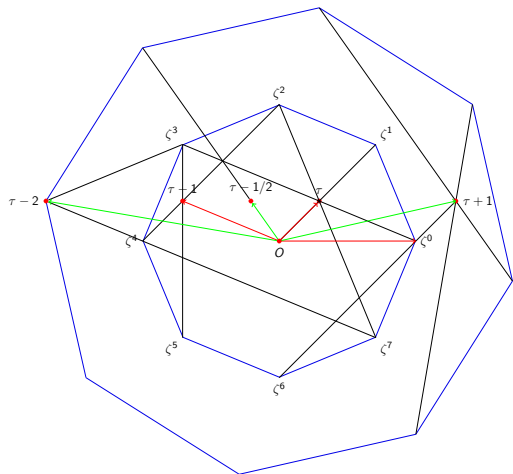


Figure: $\tau = \zeta_8^2 - \zeta_8 + 1$

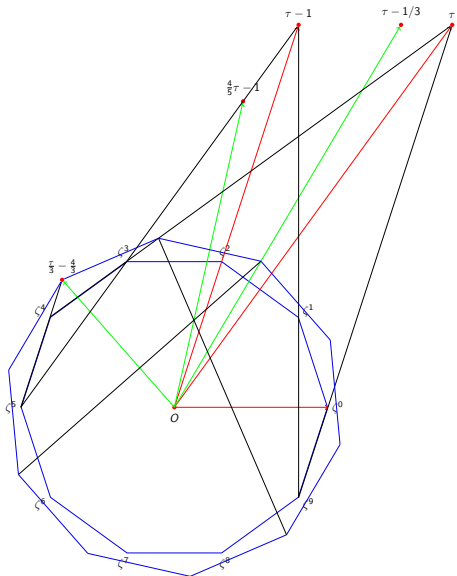


Figure: $\tau = \zeta_{10}^4 + 2\zeta_{10} + 2$

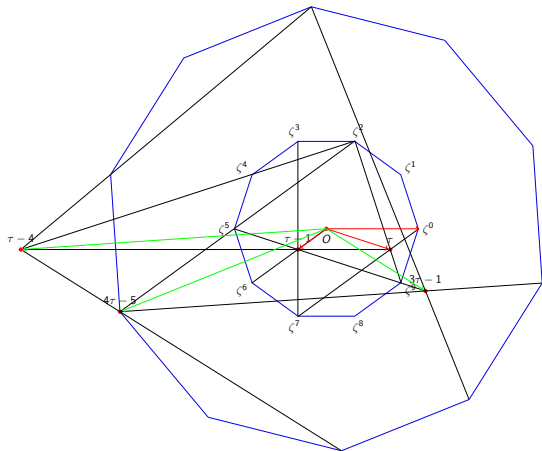


Figure: $\tau = \zeta_{10}^2 - 2\zeta_{10} + 2$

Case $ab = cd$

Two spaces of type 3(2) given by
 $(a, b, c, d) = (3, \frac{1}{2}, \frac{3}{2}, 1)$

$$\tau_1 = -\frac{1}{2}\sqrt{\frac{1}{2}\left(2\sqrt{5} + \sqrt{52\sqrt{5} + 45 + 13}\right)} \cdot e^{\frac{3\pi i}{10}}, \quad \tau_2 = -\frac{1}{2}\sqrt{\frac{1}{2}\left(2\sqrt{5} + \sqrt{52\sqrt{5} + 45 + 13}\right)} \cdot e^{\frac{3\pi i}{10}}$$

$$\arg(\tau/\bar{\tau}) = \frac{3}{10} \cdot 2\pi, \quad \arg\left(\frac{\tau+a}{\tau+b} / \frac{\bar{\tau}+a}{\bar{\tau}+b}\right) = \frac{2}{5} \cdot 2\pi, \quad \arg\left(\frac{\tau+c}{\tau+d} / \frac{\bar{\tau}+c}{\bar{\tau}+d}\right) = \frac{1}{10} \cdot 2\pi,$$

and one rational families of type 5(2) involving sixth roots of unity plus an inequality condition. Within this subfamily there is exactly one more space of type 3(2)

- One rational family of type $4(2)$ and angles $\zeta_8, \zeta_8, \zeta_8^3, \zeta_8^3$
- One elliptic family of type $6(2)$ and angles $\zeta_8, \zeta_8, \zeta_8^3, \zeta_8^3, \zeta_8^2, \zeta_8^2$
- One elliptic family of type $6(2)$ and angles $\zeta_8, \zeta_8, \zeta_8^3, \zeta_8^3, \zeta_8^4, \zeta_8^4$
- One elliptic family of type $4(2)$ and angles $\zeta_{10}, \zeta_{10}, \zeta_{10}^2, \zeta_{10}^2$
- One elliptic family of type $4(2)$ and angles $\zeta_{10}, \zeta_{10}, \zeta_{10}^3, \zeta_{10}^3$
- One elliptic family of type $4(2)$ and angles $\zeta_{10}, \zeta_{10}, \zeta_{10}^4, \zeta_{10}^4$
- One elliptic family of type $4(2)$ and angles $\zeta_{10}, \zeta_{10}, \zeta_{10}^5, \zeta_{10}^5$
- One elliptic family of type $4(2)$ and angles $\zeta_{10}^2, \zeta_{10}^2, \zeta_{10}^5, \zeta_{10}^5$
- One rational family of type $4(2)$ and angles $\zeta_{12}, \zeta_{12}, \zeta_{12}^5, \zeta_{12}^5$
- One elliptic family of type $6(2)$ and angles $\zeta_{12}, \zeta_{12}, \zeta_{12}^5, \zeta_{12}^5, \zeta_{12}^4, \zeta_{12}^4$
- One elliptic family of type $6(2)$ and angles $\zeta_{12}, \zeta_{12}, \zeta_{12}^5, \zeta_{12}^5, \zeta_{12}^6, \zeta_{12}^6$
- One elliptic family of type $4(2)$ and angles $\zeta_{12}^3, \zeta_{12}^3, \zeta_{12}^2, \zeta_{12}^2$
- One elliptic family of type $4(2)$ and angles $\zeta_{12}^3, \zeta_{12}^3, \zeta_{12}^4, \zeta_{12}^4$

Description of the families in $\mathbb{Q}(\zeta_8)$

For $(\lambda, \mu) \in \mathbb{Q}^2 \setminus \{(0, 0)\}$, let

$$\tau_{\lambda, \mu} := \lambda(\zeta_8^2 - \zeta_8^3) + \mu(1 + \zeta_8) = \lambda \cdot \sqrt{2 - \sqrt{2}}\zeta_{16} + \mu \cdot \sqrt{2 + \sqrt{2}}\zeta_{16}.$$

The following hold:

- The space $\langle 1, \tau_{\lambda, \mu} \rangle_{\mathbb{Q}}$ only depends on $[\lambda : \mu] \in \mathbb{P}_1(\mathbb{Q})$. We denote it by $V_{[\lambda : \mu]}$.
- Let $\tau \in \mathbb{Q}(\zeta_8)$ generate a space V such that the squared angle between 1 and τ is ζ_8 . There exists $(\lambda, \mu) \in \mathbb{Q}^2 \setminus \{(0, 0)\}$ such that $\tau = \tau_{\lambda, \mu}$ and conversely, for every $(\lambda, \mu) \in \mathbb{Q}^2 \setminus \{(0, 0)\}$, the squared angle between 1 and τ is ζ_8 .
- For every $[\lambda : \mu] \in \mathbb{P}_1(\mathbb{Q})$, the complex conjugate $\overline{V_{[\lambda : \mu]}}$ is homothetic to $V_{[\lambda : \mu]}$.

Description of the families in $\mathbb{Q}(\zeta_8)$

- The map

$$\begin{aligned} f : \mathbb{P}_1(\mathbb{Q}) &\rightarrow \{\text{homothety classes of spaces}\} \\ [\lambda : \mu] &\mapsto \text{class of } V_{[\lambda:\mu]} \end{aligned}$$

is 2-to-1. The involution $\iota : [\lambda : \mu] \mapsto [\lambda + \mu : \lambda - \mu]$ preserves the fibres of f .

- Let $S = \{[0 : 1], [1 : 0], [1 : 1], [-1 : 1]\}$. The spaces corresponding to $[\lambda : \mu] \in S$ have types $\textcircled{4}$ or $2\textcircled{3}$.

Description of the families in $\mathbb{Q}(\zeta_8)$

- If $[\lambda : \mu] \notin S$, the space $V_{[\lambda:\mu]}$ contains precisely two angles with squared amplitude ζ_8 and precisely two angles with squared amplitude ζ_8^3 . These are given respectively by

$$\zeta_8 : (1, \tau) \text{ and } (\tau + a, \tau + b),$$

where

$$a = \frac{\lambda^2 - 2\lambda\mu - \mu^2}{\lambda + \mu}, \quad b = -\frac{2(\lambda^3 + 2\lambda^2\mu - \lambda\mu^2)}{\lambda^2 - 2\lambda\mu - \mu^2}$$

and by

$$\zeta_8^3 : (\tau + c, \tau + d) \text{ and } (\tau + c', \tau + d')$$

where

$$c = \frac{-\lambda^2 - 2\lambda\mu + \mu^2}{2\lambda}, \quad d = -2(\lambda + \mu)$$

$$c' = -2\mu, \quad d' = \frac{\lambda^2 + 2\lambda\mu - \mu^2}{\mu - \lambda}.$$

Description of the families in $\mathbb{Q}(\zeta_8)$

- Let $[\lambda : \mu] \notin S$. The space $V_{[\lambda:\mu]}$ contains an angle with squared amplitude ζ_8^2 if and only if $\lambda(\lambda^3 + \lambda^2\mu + \lambda\mu^2 + \mu^3)$ is the square of a rational number u . In this case, setting

$$e_{\pm} = \frac{\lambda^2 - \mu^2 \mp u}{\mu}, \quad f_{\pm} = \frac{-2\lambda\mu \pm u}{\lambda},$$

the space $V_{[\lambda:\mu]}$ contains precisely two angles with squared amplitude ζ_8^2 , given by $(\tau + e_+, \tau + f_+)$ and $(\tau + e_-, \tau + f_-)$. The curve in weighted projective space

$$u^2 = \lambda(\lambda^3 + \lambda^2\mu + \lambda\mu^2 + \mu^3)$$

has genus 1 and a rational point, so it defines an elliptic curve. It is in particular isomorphic over \mathbb{Q} to the elliptic curve with Weierstrass equation $y^2 = x^3 + x^2 + x + 1$, whose Mordell-Weil group over \mathbb{Q} is $\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. A generator for the free part of the Mordell-Weil group is given by $(0, 1)$.

Description of the families in $\mathbb{Q}(\zeta_8)$

- Let $[\lambda : \mu] \notin S$. The space $V_{[\lambda:\mu]}$ contains an angle with squared amplitude ζ_8^4 if and only if $3\lambda^4 + 2\lambda^3\mu + 4\lambda^2\mu^2 + 2\lambda\mu^3 + \mu^4$ is the square of a rational number v . In this case, setting

$$g = \frac{\lambda^2 - 2\lambda\mu - \mu^2 \mp v}{\lambda + \mu}, \quad h = \frac{\lambda^2 - 2\lambda\mu - \mu^2 \pm v}{\lambda + \mu},$$

the space $V_{[\lambda:\mu]}$ contains precisely two angles with squared amplitude ζ_8^4 are precisely $(\tau + g_+, \tau + h_+)$ and $(\tau + g_-, \tau + h_-)$. The curve in weighted projective space

$$v^2 = 3\lambda^4 + 2\lambda^3\mu + 4\lambda^2\mu^2 + 2\lambda\mu^3 + \mu^4$$

has genus 1 and a rational point, so it defines an elliptic curve. It is in particular isomorphic over \mathbb{Q} to the elliptic curve with Weierstrass equation $y^2 + 2xy - 2y = x^3 - 6x^2 - 3x$, whose Mordell-Weil group over \mathbb{Q} is $\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. A generator for the free part of the Mordell-Weil group is given by $(0, 2)$.

- Let $[\lambda : \mu] \notin S$. The space $V_{[\lambda:\mu]}$ does not have both an angle with squared amplitude ζ_8^2 and an angle with squared amplitude ζ_8^4 .