Rational angles in plane lattices

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Regular polygons with vertices on a plane lattice

Édouard Lucas, 1878

Les centres de trois cases quelconques d'un échiquier de grandeur quelconque ne sont jamais situés aux sommets d'un triangle équilatéral ou d'un hexagone régulier.

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Willy Scherrer, 1946

Es existiert kein einfaches oder sich selbst durchdringendes reguläres n-Eck, dessen Ecken sämtlich Punkte eines ebenen Gitters sind, ausgenommen in den Fällen $n = 3, 4$ und 6.

There is no regular n-agon (also self-intersecting) whose vertices all belong to a plane lattice, except for the cases $n = 3, 4, 6$.

Rational amplitudes in a plane lattice

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Example in the Gaussian integers: if $P, Q \in \mathbb{Z}[i]$ and $P\hat{O}Q$ is a rational multiple of π, then

$$\begin{aligned} P/Q &= \zeta r, \quad \zeta \in U, r \in \mathbb{R} \\ \overline{P}/\overline{Q} &= \zeta^{-1} r \\ \zeta^2 &= P\overline{Q}/\overline{P}Q \in \mathbb{Q}(i). \end{aligned}$$

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This proves that if three points of $\mathbb{Z}[i]$ determine an angle with amplitude in $\pi\mathbb{Q}$, then this amplitude is in fact in $\frac{\pi}{4}\mathbb{Z}$; this generalizes Lucas's result.

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To the purposes of this classification it is convenient to replace a lattice $\Lambda \subset \mathbb{C}$ with the $\mathbb{Q}$-vector space $V = \Lambda \otimes_{\mathbb{Z}} \mathbb{Q} \subset \mathbb{C}$.
We study these spaces up to homotheties and complex conjugation.

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Two angles $\rightsquigarrow$ one equation
If $\theta, \mu \in U$, $\tau = \theta r$, $a, b \in \mathbb{Q}$, $ab(a-b) \neq 0$ and

$$\mu \frac{\tau + a}{\tau + b} \in \mathbb{R},$$

we obtain

$$r^2 + r \frac{a(y-x) + b(xy-1)}{\theta(y-1)} + ab = 0,$$

$\theta^2 = x, \mu^2 = y$.

Rational n-tuples

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We call *rational n-tuple* a set of n vectors such that each pair of them forms a rational angle.
We say that a space has type $\textcircled{n}$ if it contains a rational n-tuple; we extend this notation additively, saying that it has type $\textcircled{n}+\textcircled{m}$ if is contains a rational n-tuple and a disjoint rational m-tuple, and so on.

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Given two *independent* equations as before we can eliminate r

$$P_{④}(a,b,x,y,z) = (a-b)x + by - az - axy + bxz + (a-b)yz = 0.$$

$$\begin{aligned} P_{③+②}(a,b,c,x,y,z) = {} & (2a^2 - a(b+c) + 2bc)xy - abx^2y - acy - a(a-b)xy^2 \\ & - a(a-c)x + b(a-c)x^2 + c(a-b)y^2 \\ & - (2a^2 - a(b+c) + 2bc)xyz + acx^2yz \\ & + abyz + a(a-c)xy^2z + a(a-b)xz \\ & - c(a-b)x^2z - b(a-c)y^2z = 0. \end{aligned}$$

The equation of three non-adjacent angles

$$
\begin{aligned}
&P_{3\textcircled{2}}(a,b,c,d,x,y,z) = \\
&-b(b-c)c(a-d)y^2x^2 - b(a-c)(b-d)dy^2z^2x^2 - a(b-c)(a-d)dz^2x^2 + d\left(ba^2+b^2a-2bca-2bda+cda+bcd\right)yz^2x^2 \\
&-a(a-c)c(b-d)x^2 + c\left(ba^2+b^2a-2bca-2bda+cda+bcd\right)yx^2 + b\left(dc^2+d^2c+abc-2adc-2bdc+abd\right)y^2zx^2 \\
&+a\left(dc^2+d^2c+abc-2adc-2bdc+abd\right)zx^2 + \left(-bca^2-bda^2-cd^2a-b^2ca-b^2da-c^2da+8bcda-bcd^2-bc^2d\right)yzx^2 \\
&+(b-c)(a-d)(ab+cd)y^2x + (a-c)(b-d)(ab+cd)y^2z^2x + (b-c)(a-d)(ab+cd)z^2x \\
&+\left(-2b^2a^2+bca^2+bda^2-2cda^2+cd^2a+b^2ca+b^2da+c^2da-2c^2d^2+bcd^2+bc^2d-2b^2cd\right)yz^2x \\
&+(a-c)(b-d)(ab+cd)x + \left(-2b^2a^2+bca^2+bda^2-2cda^2+cd^2a+b^2ca+b^2da+c^2da-2c^2d^2+bcd^2+bc^2d-2b^2cd\right)yx \\
&+\left(-2b^2a^2+bca^2+bda^2-2bc^2a-2bd^2a+cd^2a+b^2ca+b^2da+c^2da-2c^2d^2+bcd^2+bc^2d\right)y^2zx \\
&+\left(-2b^2a^2+bca^2+bda^2-2bc^2a-2bd^2a+cd^2a+b^2ca+b^2da+c^2da-2c^2d^2+bcd^2+bc^2d\right)zx \\
&+2\left(2b^2a^2-bca^2-bda^2+2cda^2+2bc^2a+2bd^2a-cd^2a-b^2ca-b^2da-c^2da-4bcda+2c^2d^2-bcd^2-bc^2d+2b^2cd\right)yzx \\
&-a(b-c)(a-d)dy^2 - a(a-c)c(b-d)y^2z^2 - b(b-c)c(a-d)z^2 + c\left(ba^2+b^2a-2bca-2bda+cda+bcd\right)yz^2 \\
&-b(a-c)(b-d)d + d\left(ba^2+b^2a-2bca-2bda+cda+bcd\right)y + a\left(dc^2+d^2c+abc-2adc-2bdc+abd\right)y^2z \\
&+b\left(dc^2+d^2c+abc-2adc-2bdc+abd\right)z + \left(-bca^2-bda^2-cd^2a-b^2ca-b^2da-c^2da+8bcda-bcd^2-bc^2d\right)yz = 0
\end{aligned}
$$

we seek solutions with x, y, z roots of unity different from 1 and a, b, c, d rational numbers, all distinct and different from 0.

When independent angles lead to proportional equations

$$P_{ab=cd} = (d-b)xyz + (b-c)xy + (a-d)xz + (c-a)yz + \\ + (c-a)x + (a-d)y + (b-c)z + (d-b) = 0.$$

But in this case, unlike the others, an algebraic solution doesn't necessarily lead to a geometric configuration.

Diophantine-trigonometric equations

Gordan in 1877 studied the equation

$$\cos\alpha + \cos\beta + \cos\gamma = -1$$

to be solved in rational angles in order to classify finite subgroups of PGL_2.

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Theorem (Conway-Jones, 1976)

Let

$$\sum_{j=0}^{k-1} a_j \xi_j = 0$$

be a linear relation with rational coefficients a_j among roots of unity ξ_j, normalized so that $\xi_0 = 1$.
Then, either there is a proper subset of the summands which sums to zero, or the common order Q of the ξ_j is a squarefree number such that

$$\sum_{p|Q} (p-2) \leq k-2.$$

An approach

The problem consists in solving four diophantine trigonometric equations.
A potential approach consists in applying iteratively the Conway-Jones theorem on all sub-sums to reach a bound on the order of the roots of unity involved; then, for every choice of x, y, z, look for rational solutions in a, b, c, d.

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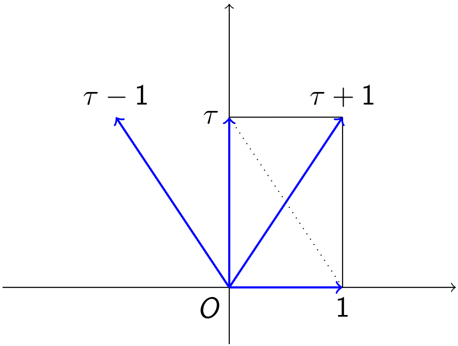
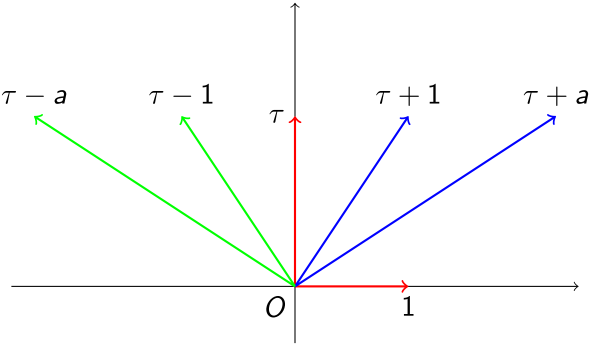
• 2^{27} sub-sums to consider
• we end up looking for rational points on curves of high genus
Successive reductions to bring down the order of the roots of unity involved:
• Minkowski's theorem
• Shortest vector computations on finitely many lattices
• Excluding certain prime divisors from the common order of x, y, z by considering the action of part of the Galois group of the extension $\mathbb{Q}(x, y, z)/\mathbb{Q}$.

Families with unbounded order (1)



A family of type ④

Families with unbounded order (2)



For every $a > 0$ in $\mathbb{Q}$, this is a family of type 3②

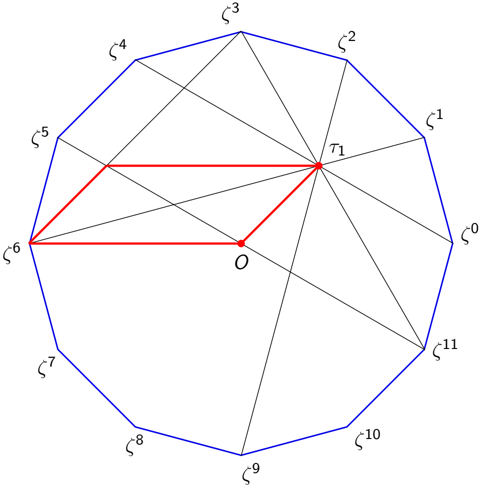
Spaces with infinitely many rational angles

The spaces $\mathbb{Q}(\sqrt{-d})$ with $d > 0$ squarefree integer are examples of infinitely many spaces, all not homotetic one another, each containing infinitely many rational angles.

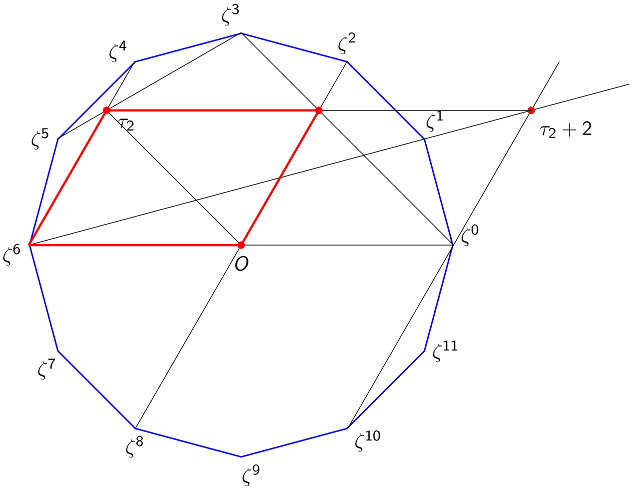
- for $d = 1$, $\mathbb{Q}(\sqrt{-d})$ has type ∞④
- for $d = 3$, $\mathbb{Q}(\sqrt{-d})$ has type ∞⑥
- for $d \neq 1, 3$, $\mathbb{Q}(\sqrt{-d})$ has type ∞②

Dodecagonal spaces

There are exactly two more spaces of type (4) up to homothety



dodecagonal spaces



Type ③+②

There are exactly four spaces of type 2③:

- $\langle \zeta_8^2 + \zeta_8 + 1, 1 \rangle_{\mathbb{Q}}$
- $\langle \zeta_5^2 + 2\zeta_5 + 2, 1 \rangle_{\mathbb{Q}}$
- $\langle 2\zeta_5^3 + \zeta_5 + 2, 1 \rangle_{\mathbb{Q}}$
- $\langle 2\zeta_{12}^3 + 2\zeta_{12}^2 - \zeta_{12} - 1, 1 \rangle_{\mathbb{Q}}$

they also correspond to intersection of diagonals in regular polygons.

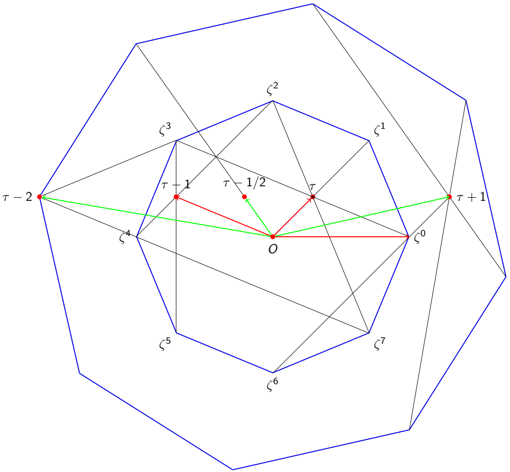


Figure: $\tau = \zeta_8^2 - \zeta_8 + 1$

Case $ab = cd$

Two spaces of type $3\textcircled{2}$ given by
$(a, b, c, d) = (3, \frac{1}{2}, \frac{3}{2}, 1)$

$$\tau_1 = -\frac{1}{2}\sqrt{\frac{1}{2}\left(2\sqrt{5} + \sqrt{52\sqrt{5} + 45} + 13\right)} \cdot e^{\frac{3\pi i}{10}}, \quad \tau_2 = -\frac{1}{2}\sqrt{\frac{1}{2}\left(2\sqrt{5} + \sqrt{52\sqrt{5} + 45} + 13\right)} \cdot e^{\frac{3\pi i}{10}}$$

$$\arg(\tau/\overline{\tau}) = \frac{3}{10}\cdot 2\pi, \quad \arg\left(\frac{\tau + a}{\tau + b} \Big/ \frac{\overline{\tau} + a}{\overline{\tau} + b}\right) = \frac{2}{5}\cdot 2\pi, \quad \arg\left(\frac{\tau + c}{\tau + d} \Big/ \frac{\overline{\tau} + c}{\overline{\tau} + d}\right) = \frac{1}{10}\cdot 2\pi,$$

and one more rational families of type $3\textcircled{2}$ involving sixth roots of unity plus an inequality condition.

One rational family of type $4\textcircled{2}$ and angles $\zeta_8, \zeta_8, \zeta_8^3, \zeta_8^3$

One elliptic family of type $6\textcircled{2}$ and angles $\zeta_8, \zeta_8, \zeta_8^3, \zeta_8^3, \zeta_8^2, \zeta_8^2$

One elliptic family of type $6\textcircled{2}$ and angles $\zeta_8, \zeta_8, \zeta_8^3, \zeta_8^3\zeta_8^4, \zeta_8^4$

One elliptic family of type $4\textcircled{2}$ and angles $\zeta_{10}, \zeta_{10}, \zeta_{10}^2, \zeta_{10}^2$

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One elliptic family of type $4\textcircled{2}$ and angles $\zeta_{10}^2, \zeta_{10}^2, \zeta_{10}^5, \zeta_{10}^5$

One rational family of type $4\textcircled{2}$ and angles $\zeta_{12}, \zeta_{12}, \zeta_{12}^5, \zeta_{12}^5$

One elliptic family of type $6\textcircled{2}$ and angles $\zeta_{12}, \zeta_{12}, \zeta_{12}^5, \zeta_{12}^5, \zeta_{12}^4, \zeta_{12}^4$

One elliptic family of type $6\textcircled{2}$ and angles $\zeta_{12}, \zeta_{12}, \zeta_{12}^5, \zeta_{12}^5, \zeta_{12}^6, \zeta_{12}^6$

One elliptic family of type $4\textcircled{2}$ and angles $\zeta_{12}^3, \zeta_{12}^3, \zeta_{12}^2, \zeta_{12}^2$

One elliptic family of type $4\textcircled{2}$ and angles $\zeta_{12}^3, \zeta_{12}^3, \zeta_{12}^4, \zeta_{12}^4$

Description of the familes in $\mathbb{Q}(\zeta_8)$

For $(\lambda, \mu) \in \mathbb{Q}^2 \setminus \{(0,0)\}$, let

$$\tau_{\lambda,\mu} := \lambda(\zeta_8^2 - \zeta_8^3) + \mu(1+\zeta_8) = \lambda \cdot \sqrt{2-\sqrt{2}}\zeta_{16} + \mu \cdot \sqrt{2+\sqrt{2}}\zeta_{16}.$$

The following hold:

- The space $\langle 1, \tau_{\lambda,\mu} \rangle_{\mathbb{Q}}$ only depends on $[\lambda : \mu] \in \mathbb{P}_1(\mathbb{Q})$. We denote it by $V_{[\lambda:\mu]}$.
- Let $\tau \in \mathbb{Q}(\zeta_8)$ generate a space V such that the squared angle between 1 and τ is ζ_8. There exists $(\lambda, \mu) \in \mathbb{Q}^2 \setminus \{(0,0)\}$ such that $\tau = \tau_{\lambda,\mu}$ and conversely, for every $(\lambda, \mu) \in \mathbb{Q}^2 \setminus \{(0,0)\}$, the squared angle between 1 and τ is ζ_8.
- For every $[\lambda : \mu] \in \mathbb{P}_1(\mathbb{Q})$, the complex conjugate $\overline{V_{[\lambda:\mu]}}$ is homothetic to $V_{[\lambda:\mu]}$.

Description of the familes in $\mathbb{Q}(\zeta_8)$

- The map

$$f: \begin{array}{ccc} \mathbb{P}_1(\mathbb{Q}) & \to & \{\text{homothety classes of spaces}\} \\ [\lambda:\mu] & \mapsto & \text{class of } V_{[\lambda:\mu]} \end{array}$$

  is 2-to-1. The involution $\iota : [\lambda:\mu] \mapsto [\lambda+\mu : \lambda-\mu]$ preserves the fibres of f.

- Let $S = \{[0:1],[1:0],[1:1],[-1:1]\}$. The spaces corresponding to $[\lambda:\mu] \in S$ have types $\textcircled{4}$ or $2\textcircled{3}$.

Description of the familes in $\mathbb{Q}(\zeta_8)$

- If $[\lambda : \mu] \notin S$, the space $V_{[\lambda:\mu]}$ contains precisely two angles with squared amplitude ζ_8 and precisely two angles with squared amplitude ζ_8^3. These are given respectively by

$$\zeta_8 : \quad (1, \tau) \text{ and } (\tau + a, \tau + b),$$

where

$$a = \frac{\lambda^2 - 2\lambda\mu - \mu^2}{\lambda + \mu}, \quad b = -\frac{2\left(\lambda^3 + 2\lambda^2\mu - \lambda\mu^2\right)}{\lambda^2 - 2\lambda\mu - \mu^2}$$

and by

$$\zeta_8^3 : \quad (\tau + c, \tau + d) \text{ and } (\tau + c', \tau + d')$$

where

$$c = \frac{-\lambda^2 - 2\lambda\mu + \mu^2}{2\lambda}, \quad d = -2(\lambda + \mu)$$

$$c' = -2\mu, \quad d' = \frac{\lambda^2 + 2\lambda\mu - \mu^2}{\mu - \lambda}.$$

Description of the familes in $\mathbb{Q}(\zeta_8)$

- Let $[\lambda : \mu] \notin S$. The space $V_{[\lambda:\mu]}$ contains an angle with squared amplitude ζ_8^2 if and only if $\lambda(\lambda^3 + \lambda^2\mu + \lambda\mu^2 + \mu^3)$ is the square of a rational number u. In this case, setting

$$e_{\pm} = \frac{\lambda^2 - \mu^2 \mp u}{\mu}, \quad f_{\pm} = \frac{-2\lambda\mu \pm u}{\lambda},$$

the space $V_{[\lambda:\mu]}$ contains precisely two angles with squared amplitude ζ_8^2, given by $(\tau + e_+, \tau + f_+)$ and $(\tau + e_-, \tau + f_-)$. The curve in weighted projective space

$$u^2 = \lambda(\lambda^3 + \lambda^2\mu + \lambda\mu^2 + \mu^3)$$

has genus 1 and a rational point, so it defines an elliptic curve. It is in particular isomorphic over $\mathbb{Q}$ to the elliptic curve with Weierstrass equation $y^2 = x^3 + x^2 + x + 1$, whose Mordell-Weil group over $\mathbb{Q}$ is $\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. A generator for the free part of the Mordell-Weil group is given by $(0, 1)$.

Description of the familes in $\mathbb{Q}(\zeta_8)$

- Let $[\lambda : \mu] \notin S$. The space $V_{[\lambda:\mu]}$ contains an angle with squared amplitude ζ_8^4 if and only if $3\lambda^4 + 2\lambda^3\mu + 4\lambda^2\mu^2 + 2\lambda\mu^3 + \mu^4$ is the square of a rational number v. In this case, setting

$$g = \frac{\lambda^2 - 2\lambda\mu - \mu^2 \mp v}{\lambda + \mu}, \quad h = \frac{\lambda^2 - 2\lambda\mu - \mu^2 \pm v}{\lambda + \mu},$$

  the space $V_{[\lambda:\mu]}$ contains precisely two angles with squared amplitude ζ_8^4 are precisely $(\tau + g_+, \tau + h_+)$ and $(\tau + g_-, \tau + h_-)$. The curve in weighted projective space

$$v^2 = 3\lambda^4 + 2\lambda^3\mu + 4\lambda^2\mu^2 + 2\lambda\mu^3 + \mu^4$$

  has genus 1 and a rational point, so it defines an elliptic curve. It is in particular isomorphic over $\mathbb{Q}$ to the elliptic curve with Weierstrass equation $y^2 + 2xy - 2y = x^3 - 6x^2 - 3x$, whose Mordell-Weil group over $\mathbb{Q}$ is $\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. A generator for the free part of the Mordell-Weil group is given by $(0, 2)$.
- Let $[\lambda : \mu] \notin S$. The space $V_{[\lambda:\mu]}$ does not have both an angle with squared amplitude ζ_8^2 and an angle with squared amplitude ζ_8^4.