

# Finiteness and periodicity for $\beta$ -continued fractions

joint work in progress with Zuzana Másáková and Tomáš Vávra

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# Continued Fractions

A continued fraction is an expression of the form

$$[a_0, a_1, \dots, a_n] := a_0 + \frac{1}{a_1 + \frac{1}{\dots + \frac{1}{a_{n-1} + \frac{1}{a_n}}}} = \frac{p_n}{q_n}$$

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In the classical setting, we take the  $a_i$  to be positive integers. In this case it makes sense to consider an infinite sequence of  $a_i$ 's and the corresponding limit of the values  $p_n/q_n$ .

# Continued Fraction Expansion

Starting from a real number  $\alpha = \alpha_0$  we define the iteration

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and the recurrences

$$p_n = a_n p_{n-1} + p_{n-2}, \quad p_{-1} = 1, p_{-2} = 0,$$
$$q_n = a_n q_{n-1} + q_{n-2}, \quad q_{-1} = 0, q_{-2} = 1.$$

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The  $a_n$  are called *partial quotients*

The  $\alpha_n$  are called *complete quotients*

The  $p_n/q_n$  are called *convergents*

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The expansion is eventually periodic if and only if  $\alpha$  is a quadratic irrational (Lagrange)

The convergents provide very good rational approximations to  $\alpha$ .

# A Question of Rosen

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Is it possible to devise a continued fraction that represents uniquely all real numbers, so that the finite continued fractions represent the elements of an algebraic number field, and conversely, every element of the number field is represented by a finite continued fraction?

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Rosen gives one example of such a construction for the field  $\mathbb{Q}(\sqrt{5})$  and partial quotients which are integral multiples of  $\phi = \frac{1+\sqrt{5}}{2}$ .  
Bernat '06 gives a different construction again for  $\mathbb{Q}(\sqrt{5})$ .

# $\beta$ -expansions

Let  $\beta > 1$  be an algebraic integer. Any real number  $x$  can be expanded in base- $\beta$  as

$$x = \pm \sum_{i=-\infty}^k x_i \beta^i.$$

The digits  $x_i$  belong to the set  $\{0, 1, \dots, \lceil \beta \rceil - 1\}$ , and are selected according to a greedy algorithm.

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Not all expansions are admissible.  $\phi^2 = \phi + 1$

# $\beta$ -integers

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For some special  $\beta$ 's e.g. for Pisot numbers, it is possible to give an algebraic characterization of this set in terms of their algebraic conjugates.

## $\beta$ -fractionary expansion

For a positive real number  $x$ , define

$$\lfloor x \rfloor_{\beta} = \max\{a \in \mathbb{Z}_{\beta} \mid a \leq x\}.$$

Replace  $\lfloor \cdot \rfloor$  by  $\lfloor \cdot \rfloor_{\beta}$  in the definition of the continued fraction expansion.

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### Question (Bernat '06)

For which other numbers (quadratic Pisot) does the same conclusion hold?

# Periodicity and finiteness for the $\beta$ -fractionary expansion

Let  $\beta > 1$  be an algebraic integer.

(CFP)

We say that  $\beta$  has the (CFP) property if the  $\beta$ -fractionary expansion of every element of  $\mathbb{Q}(\beta)$  is finite or eventually periodic.

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## Results on quadratic $\beta$ 's

From now on, let  $\beta > 1$  be a quadratic integer, and let  $\beta'$  be its algebraic conjugate.

### Theorem (Másáková, V, Vávra)

*If  $|\beta'| < \beta$  (Perron numbers), then (CFP) holds.*

*Every purely periodic element in  $\mathbb{Q}(\beta)$  has partial quotients in  $\{1, \dots, \lfloor \beta \rfloor\}$ .*

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We use an argument of diophantine approximation and a comparison lemma to estimate the relative growths of the sequences  $p_n, q_n$  and their conjugates.

## Theorem (Másáková, Vávra, V)

*The four Perron numbers*

$$\frac{1 + \sqrt{5}}{2}, \quad 1 + \sqrt{2}, \quad \frac{1 + \sqrt{13}}{2}, \quad \frac{1 + \sqrt{17}}{2}$$

*have (CFF), and are the only quadratic Perron numbers smaller than 3 with property (CFF).*

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We apply the previous theorem and argue about admissible sequences of partial quotients.

$$\frac{164 + 65\sqrt{17}}{251} = [1, 1, 2, 1, 1, 2, 2, 2, 2]$$

$$\frac{164 + 65\sqrt{17}}{251} = [1, 1, \beta, 2\beta^3 + \beta^2 + 1, \beta^3 + \beta + 1, 2, \beta + 1]$$

## Conjecture (McMullen '08)

*Every real quadratic number field contains infinitely many elements whose (classical) continued fraction expansion consists only in 1's and 2's.*

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### Conjecture (Mercat '13)

*Every real quadratic number field contains one element whose (classical) continued fraction expansion consists only in 1's and 2's.*

Under Mercat's conjecture, no quadratic  $\beta > 3$  can have property (CFF).

### Theorem (Másáková, Vávra, V)

*If  $\beta' > \beta$ , then (CFP) never holds.*

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*If  $\beta' > \beta$ , then (CFP) never holds.*

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We use an argument of algebraic number theory and a characterization of pure periodicity.